Control of underactuated systems – from theory to practice *

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Schedule

Overview

Mathematical Properties of the Model

Consider the system with multiple inputs with dynamics given by:

$$\dot{x} = f(x) + G(x)u, \tag{1}$$

where

$$f(x) = \begin{bmatrix} f_1(x) \\ \vdots \\ f_n(x) \end{bmatrix}, \; G(x) = \begin{bmatrix} g_{11}(x) & \cdots & g_{1m}(x) \\ \vdots & \ddots & \vdots \\ g_{n1}(x) & \cdots & g_{nm}(x) \end{bmatrix} = \begin{bmatrix} g_1(x) & \cdots & g_m(x) \end{bmatrix}. \tag{2}$$

where $x \in \mathbb{R}^n, \; u \in \mathbb{R}^m$ and $f(x), G(x)$ are of appropriate dimensions.

We can define the distributions

$$\boldsymbol{D}_j = \operatorname{span}\{g_1, ..., g_m, ad_f g_1, ..., ad_f g_m, ..., ad_f^{j-1} g_1, ..., ad_f^{j-1} g_m\}$$

where:
$ad_f g_i = [f, g_i] = \frac{\partial g_i}{\partial x} f - \frac{\partial f}{\partial x} g_i$, for any $k \geq 1$, setting $ad_f^0 g_i(x) = g_i(x)$,
and let $\bar{\boldsymbol{D}}_j$ denote the involutive closure of $\boldsymbol{D}_j$, which is the smallest involutive distribution containing $\boldsymbol{D}_j$ and $j = 0, 1, ..., n-1$.

* Distribution D in involutive if the Lie Bracket $[f_i(x), f_j(x)]$ for any pair of vector fields $f_i(x), f_j(x)$ belonging to D is a vector field which belongs to D.

Mathematical Properties of the Model

Aim

1. analyze properties of underactuated 3 DOF pendulum
2. stabilize it in vertical position

- When the system is underactuated, full feedback linearisation is not possible.
- The system should be decomposed into two subsystems, one which is linear, and one which stays still nonlinear.
- An important issue is the maximal dimension of the linear subsystem that might be obtained.

Overview

Robot Model

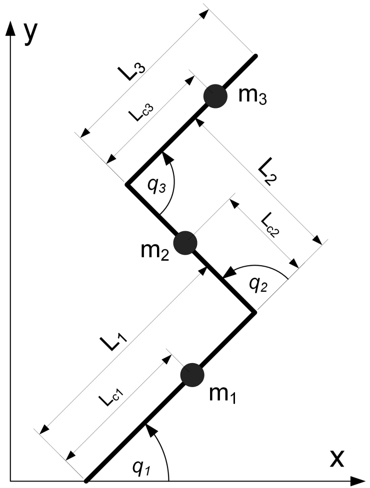


Figure 1: 3-link pendulum

3-link robot

- $N = 3$ rigid bodies coupled in a tree structure
- supported on ground via an actuated frictionless revolute joint
- one degree of underactuation (3 DOF with 2 independent actuators)

Table 1: Robot parameters

m_i – Mass $[\mathrm{kg}]$	Centre of mass $[\mathrm{m}]$	L_i – Length $[\mathrm{m}]$	Inertia $[\mathrm{kg\,m^2}]$
1.118	0.062	0.07	0.0118
1.593	0.074	0.15	0.0119
0.405	0.134	0.295	0.0117

Robot Model

In order to establish the system dynamics one can define Lagrangian

$$L = K - V$$

while $K = \frac{1}{2}\dot{q}^T M(q)\dot{q}$ denotes kinetic energy, with M being a positive definite inertia matrix, and V is the potential energy.
Next, taking into account the actuation on the system (Fig. 2) one obtains

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_k} - \frac{\partial L}{\partial q_k} = \begin{cases} \tau_k, & k = 1,2 \\ 0, & k = 3 \end{cases} \tag{3}$$

with $\tau_k \in \mathbb{R}$.

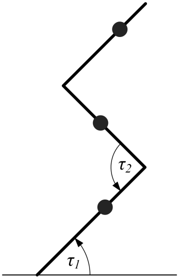


Figure 2: Triple pendulum – underactuated model

Robot Model

The overall model of dynamics can be written in a standard form of:

$$M(q)\ddot{q} + C(q,\dot{q})\dot{q} + G(q) = \tau \tag{4}$$

where matrices M, C, G are as following:

$$M = \begin{bmatrix} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \\ m_{31} & m_{32} & m_{33} \end{bmatrix}, \quad C = \begin{bmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{bmatrix}, \quad G = \begin{bmatrix} G_1 \\ G_2 \\ G_3 \end{bmatrix}, \quad \tau = \begin{bmatrix} \tau_1 \\ \tau_2 \\ \tau_3 \end{bmatrix}, \tag{5}$$

or in equivalent form:

$$\begin{aligned} m_{11}\ddot{q}_1 + m_{12}\ddot{q}_2 + m_{13}\ddot{q}_3 + \mu_1 + G_1 &= \tau_1 \\ m_{21}\ddot{q}_1 + m_{22}\ddot{q}_2 + m_{23}\ddot{q}_3 + \mu_2 + G_2 &= \tau_2 \\ m_{31}\ddot{q}_1 + m_{32}\ddot{q}_2 + m_{33}\ddot{q}_3 + \mu_3 + G_3 &= 0 \end{aligned} \tag{6}$$

where:

$$\begin{aligned} \mu_1 &= c_{11}\dot{q}_1 + c_{12}\dot{q}_2 + c_{13}\dot{q}_3, \\ \mu_2 &= c_{21}\dot{q}_1 + c_{22}\dot{q}_2 + c_{23}\dot{q}_3, \\ \mu_3 &= c_{31}\dot{q}_1 + c_{32}\dot{q}_2 + c_{33}\dot{q}_3. \end{aligned}$$

Matrix M

The elements of the M mass matrix are as follows:

$$\begin{aligned}
m_{11} &= a_1 + a_2 + a_3 + a_4 + a_5 + 2(r_1 + r_2 + r_3) \\
m_{12} &= a_2 + a_3 + a_4 + r_1 + r_2 + 2r_3 \\
m_{13} &= a_3 + r_1 + r_3 \\
m_{21} &= m_{12} \\
m_{22} &= a_2 + a_3 + a_4 + 2r_3 \\
m_{23} &= a_3 + r_3 \\
m_{31} &= m_{13} \\
m_{32} &= m_{23} \\
m_{33} &= a_3
\end{aligned} \tag{7}$$

where

$$\begin{aligned}
a_1 &= m_1 L_{c1}^2 + I_1 \\
a_2 &= m_2 L_{c2}^2 + I_2 \\
a_3 &= m_3 L_{c3}^2 + I_3 \\
a_4 &= m_3 L_2^2 \\
a_5 &= (m_2 + m_3) L_1^2
\end{aligned} \quad , \quad
\begin{aligned}
r_1 &= L_1 L_{c3} m_3 \cos(q_2 + q_3) \\
r_2 &= L_1 (L_2 m_3 + L_{c2} m_2) \cos q_2 \\
r_3 &= L_2 L_{c3} m_3 \cos q_3 .
\end{aligned} \tag{8}$$

The Coriolis matrix C is:

$$\begin{array}{lll} c_{11} = -d_1\dot{q}_2 - d_2\dot{q}_3 & c_{12} = -d_1(\dot{q}_1 + \dot{q}_2) - d_2\dot{q}_3 & c_{13} = -d_2(\dot{q}_1 + \dot{q}_2 + \dot{q}_3) \\ c_{21} = d_1\dot{q}_1 - d_3\dot{q}_3 \quad , & c_{22} = -d_3\dot{q}_3 \quad , & c_{23} = -d_3(\dot{q}_1 + \dot{q}_2 + \dot{q}_3) \\ c_{31} = d_2\dot{q}_1 + d_3\dot{q}_2 & c_{32} = d_3(\dot{q}_1 + \dot{q}_2) & c_{33} = 0. \end{array} \tag{9}$$

with

$$\begin{array}{lcl} d_1 & = & L_1 L_{c3} m_3 \sin(q_2 + q_3) + (m_2 L_{c2} + m_3 L_2) L_1 \sin q_2 \\ d_2 & = & L_1 L_{c3} m_3 \sin(q_2 + q_3) + L_2 L_{c3} m_3 \sin q_3 \\ d_3 & = & L_2 L_{c3} m_3 \sin q_3. \end{array} \tag{10}$$

The Gravity force matrix G is as follows:

$$\begin{array}{lcl} G_1 & = & g(b_1 + b_2 + b_3) \\ G_2 & = & g(b_2 + b_3) \\ G_3 & = & g b_3 \end{array} \tag{11}$$

where:

$$\begin{array}{lcl} b_1 & = & m_1 L_{c1} \cos q_1 + (m_2 + m_3) L_1 \cos q_1 \\ b_2 & = & (m_2 L_{c2} + m_3 L_2) \cos(q_1 + q_2) \\ b_3 & = & m_3 L_{c3} \cos(q_1 + q_2 + q_3). \\ g & & \text{gravitational acceleration} \end{array} \tag{12}$$

Partial linearization conditions

Let's recall the equations of motion in the following form

$$\begin{bmatrix} m_{11} & m_{12} & m_{13} \\ m_{12} & m_{22} & m_{23} \\ m_{13} & m_{23} & m_{33} \end{bmatrix} \begin{bmatrix} \ddot{q}_1 \\ \ddot{q}_2 \\ \ddot{q}_3 \end{bmatrix} + \begin{bmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{bmatrix} \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \dot{q}_3 \end{bmatrix} + \begin{bmatrix} G_1 \\ G_2 \\ G_3 \end{bmatrix} = \begin{bmatrix} \tau_1 \\ \tau_2 \\ 0 \end{bmatrix}. \tag{13}$$

and assume that $C_1 = [c_{11}, c_{12}, c_{13}]\,\dot{q}$, $C_2 = [c_{21}, c_{22}, c_{23}]\,\dot{q}$, $C_3 = [c_{31}, c_{32}, c_{33}]\,\dot{q}$.

In the following step, we can linearize this dynamics with the use of collocated linearization

$$\ddot{q}_3 = -\frac{m_{13}\ddot{q}_1 + m_{23}\ddot{q}_2 + C_3 + G_3}{m_{33}}$$

Introduce linearizing controller:

$$\begin{aligned} \tau_1 &= \overline{m}_{11}v_1 + \overline{m}_{12}v_2 + \overline{C}_1 + \overline{G}_1 \\ \tau_2 &= \overline{m}_{21}v_1 + \overline{m}_{22}v_2 + \overline{C}_2 + \overline{G}_2 \end{aligned} \tag{14}$$

Partial linearization conditions

where

$$\begin{array}{rcl} \overline{m}_{11} & = & m_{11} + J_1 m_{31} \\ \overline{m}_{12} & = & m_{12} + J_1 m_{32} \\ \overline{m}_{21} & = & m_{21} + J_2 m_{31} \\ \overline{m}_{22} & = & m_{22} + J_2 m_{32} \end{array}, \qquad \begin{array}{rcl} \overline{C}_1 & = & C_1 + J_1 \mu_3 \\ \overline{C}_2 & = & C_2 + J_2 \mu_3 \\ \overline{G}_1 & = & G_1 + J_1 G_3 \\ \overline{G}_2 & = & G_2 + J_2 G_3. \end{array}$$

for $J_1 = -\dfrac{m_{13}}{m_{33}}$, $J_2 = -\dfrac{m_{23}}{m_{33}}$.
and v_1 i v_2 are additional control inputs, described later.

Calculations are valid when the system is not in its singularity, when:

1. $\det \begin{bmatrix} \overline{m}_{11} & \overline{m}_{12} \\ \overline{m}_{21} & \overline{m}_{22} \end{bmatrix}^{-1} = \dfrac{m_{33}}{\det M} \neq 0,$

   Here $m_{33} > 0$ and $\det M > 0$ by definition.
2. $J_1 \neq 0$ and $J_2 \neq 0$, respectively, for two cases:
   - $a_3 = m_3 L_3 (L_1 + L_2)$ for $q_2 = 0$, $q_3 = \pi + 2k\pi$;
     $a_3 < m_3 L_3 (L_1 + L_2)$ for solution of the following equation: $a_3 = -r_1 - r_3$.
   - $a_3 = m_3 L_2 L_3$ for $q_3 = \pi + 2k\pi$;
     $a_3 < m_3 L_2 L_3$ for $q_3 = -\arccos(\frac{a_3}{m_3 L_2 L_3})$.

Partial linearization conditions

In Eq. (14) variables v_1 and v_2 are new control inputs. Thus, considered system can be written in the following form

$$\begin{aligned} \ddot{q}_1 &= v_1 \\ \ddot{q}_2 &= v_2 \\ \ddot{q}_3 &= -m_{33}^{-1}(m_{31}\ddot{q}_1 + m_{32}\ddot{q}_2 + C_3 + G_3) \end{aligned} \tag{15}$$

or alternatively, introducing the state vector as:

$$x = [q_1, w_1, q_2, w_2, q_3, w_3]^\top \tag{16}$$

and substituting $C_3 + G_3 = R_3 - J_1 R_1 - J_2 R_2,$
the pendulum model is

$$\begin{aligned} \dot{q}_1 &= w_1 \\ \dot{w}_1 &= v_1 \\ \dot{q}_2 &= w_2 \\ \dot{w}_2 &= v_2 \\ \dot{q}_3 &= w_3 \\ \dot{w}_3 &= R_3 + J_1(v_1 - R_1) + J_2(v_2 - R_2). \end{aligned} \tag{17}$$

Partial linearization conditions

Using more general form, the above equation (17) can be written as:

$$\dot{x} = f(x) + g(x)u$$

or

$$\begin{bmatrix} \dot{q}_1 \\ \dot{w}_1 \\ \dot{q}_2 \\ \dot{w}_2 \\ \dot{q}_3 \\ \dot{w}_3 \end{bmatrix} = \underbrace{\begin{bmatrix} w_1 \\ 0 \\ w_2 \\ 0 \\ w_3 \\ R_3 - J_1 R_1 - J_2 R_2 \end{bmatrix}}_{f(x)} + \underbrace{\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ J_1 \end{bmatrix}}_{g_1(x)} v_1 + \underbrace{\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ J_2 \end{bmatrix}}_{g_2(x)} v_2. \tag{18}$$

where: $J_1(q_2, q_3) = -\frac{m_{13}(q_2,q_3)}{m_{33}}$, $J_2(q_3) = -\frac{m_{23}(q_3)}{m_{33}}$ and $R_i = M^{-1}(i)(-C(q,\dot{q})\dot{q} - G)$, where $M^{-1}(i)$ is an i-th row of the inverse of Mass matrix M.

Overview

Largest feedback linearizable subsystem

- As mentioned before, underactuated systems are not fully linearizable
- The question arises – what is the largest feedback linearizable subsystem of the whole system?

In order to find the largest linearizable subsystem we propose to analyze the following distributions:

- $D_0 = \mathrm{span}\{g_1, g_2\}$ – obviously is involutive
- $D_1 = \mathrm{span}\{g_1, g_2, [f, g_1], [f, g_2]\}$ – not involutive

One needs to find smallest involutive closure of D_1

- $\overline{D}_1 = \mathrm{span}\{g_1, g_2, [f, g_1], [f, g_2], [g_1, ad_f g_1]\}$ – not involutive
- $\overline{D}_1 = \mathrm{span}\{g_1, g_2, [f, g_1], [f, g_2], [g_2, ad_f g_2]\}$ – not involutive
- other combinations – not involutive
- $\overline{D}_1 = \mathrm{span}\{g_1, g_2, [f, g_1], [f, g_2], [g_1, ad_f g_2], [ad_f g_1, ad_f g_2]\}$ – involutive

Frobenius Theorem

A nosingular distribution is completely integrable if and only if is involutive.

Then one needs to find an output function h that anihilates $\overline{D}_1$, i.e.

$$\left[\frac{\partial h}{\partial x_1}\ \frac{\partial h}{\partial x_2}\ \frac{\partial h}{\partial x_3}\ \frac{\partial h}{\partial x_4}\ \frac{\partial h}{\partial x_5}\ \frac{\partial h}{\partial x_6}\right]\left[\, g_1,\ g_2,\ [f, g_1]\ \ [f, g_2], [g_1, ad_f g_1],\ [ad_f g_1, ad_f g_2]\,\right] = 0$$

As a result we get:

$$\frac{\partial h}{\partial w_1} = 0, \quad \frac{\partial h}{\partial w_2} = 0, \quad \frac{\partial h}{\partial q_1} = 0, \quad \frac{\partial h}{\partial q_2} = 0, \quad \frac{\partial h}{\partial w_3} = 0, \quad \frac{\partial h}{\partial q_3} = 0. \tag{19}$$

It is trivial that the only solution of Eq (19) is $h = \text{constant}$ because $\overline{D}_1$ is of full rank 6.

As a conclusion – the largest feedback linearizable subsystem is of dimension 4.

Largest feedback linearizable subsystem

The Lie brackets used in the above calculations are as follows:

$$\begin{array}{lcl} [f, g_1] & = & [-1\;0\;0\;0\; -J_1\; F_{16}]^\top \\ [f, g_2] & = & [0\;0\; -1\;0\; -J_2\; F_{26}]^\top \\ [g_1, ad_f g_1] & = & [0\;0\;0\;0\;0\; F_{56}]^\top \\ [g_1, ad_f g_2] & = & [0\;0\;0\;0\;0\; F_{66}]^\top \\ [g_2, ad_f g_1] & = & [0\;0\;0\;0\;0\; F_{76}]^\top \\ [g_2, ad_f g_2] & = & [0\;0\;0\;0\;0\; F_{86}]^\top \\ [ad_f g_1, ad_f g_2] & = & [0\;0\;0\;0\; F_{95}\; F_{96})]^\top , \end{array} \quad (20)$$

where:

$g_1 = [0\;1\;0\;0\;0\;J_1]^\top$

$g_2 = [0\;0\;0\;1\;0\;J_2]^\top$

$F_{16} = \frac{1}{L_3}(L_1 \sin(q_2 + q_3)[2w_1 + w_2 + w_3] + L_2 \sin(q_3)[2w_1 + 2w_2 + w_3])$

$F_{26} = \frac{1}{L_3}(L_2 \sin(q_3)(2w_1 + 2w_2 + w_3))$

$F_{56} = -\frac{1}{L_3^2}((\sin(2q_2 + 2q_3)L_1^2 + 2\sin(q_2 + 2q_3)L_1 L_2 + \sin(2q_3)L_2^2))$

$F_{66} = -\frac{1}{L_3^2}(L_1 L_2 \sin(q_2 + 2q_3) + L_2^2 \sin(2q_3))$

$F_{76} = -\frac{1}{L_3^2}(L_1 L_2 \sin(q_2 + 2q_3) + L_2^2 \sin(2q_3))$

$F_{86} = -\frac{1}{L_3^2} L_2^2 \sin(2q_3)$

$F_{95} = \frac{1}{L_3^2} L_1 L_2 \sin(q_2)$

$F_{96} = \frac{1}{L_3^2} L_1 L_2 w_2 \cos(q_2 + 2q_3)$

Overview

Stabilization problem

Aim

examine an implementation of a hybrid controller to stabilize a triple pendulum around its top unstable position, taking into account the limitations and constraints resulting from practical conditions (existing robot)

Stabilization will be obtained with the two commonly known approaches

- first – which utilizes the collocated methods for linearization
- second – the additional LQR controller is used to stabilize the system near the equilibrium point.

Stabilizing controller

$$u = \begin{cases} u_h & \text{for swing,} \\ u_{Lin} & \text{for stabilization.} \end{cases} \quad (21)$$

u_h – is used to bring the pendulum near the equilibrium pose,

$$u_h = [\tau_1, \tau_2]^\top \quad (22)$$

u_{Lin} – to stabilize at equilibrium

$$u_{Lin} = -K(x_r - x). \quad (23)$$

$x_r = [q_1^d \; q_2^d \; q_3^d \; \dot{q}_1^d \; \dot{q}_2^d \; \dot{q}_3^d \;]^\top$ and $K = \begin{bmatrix} k_1 \, k_2 \, k_3 \, k_4 & k_5 & k_6 \\ k_7 \, k_8 \, k_9 \, k_{10} & k_{11} & k_{12} \end{bmatrix}$, stand for the reference state and the controller gains, respectively.

τ_1, τ_2 are given by Eq. (14) and

$$v_1 = \ddot{q}_1 = \ddot{q}_1^d + K_1^D(\dot{q}_1^d - \dot{q}_1) + K_1^P(q_1^d - q_1) \quad (24)$$

$$v_2 = \ddot{q}_2 = \ddot{q}_2^d + K_2^D(\dot{q}_2^d - \dot{q}_2) + K_2^P(q_2^d - q_2) \quad (25)$$

where K_1^D, K_1^P, K_2^D i K_2^P are positive gains, and $q_1^d, \dot{q}_1^d, \ddot{q}_1^d, q_2^d, \dot{q}_2^d, \ddot{q}_2^d$ denote desired values at the equilibrium point.

The zero dynamics was obtained assuming that $h = const$ and

$$q_1^d = \frac{\pi}{2}, q_2^d = 0, \dot{q}_1^d = 0, \dot{q}_2^d = 0, \ddot{q}_1^d = 0, \ddot{q}_2^d = 0.$$

The resulting zero dynamics is calculated as follows

$$\ddot{q}_3 = \xi \sin q_3 \tag{26}$$

where: $\xi = \frac{1}{a_3} g m_3 L_3$, and partial solution of Eq. (26) is given by Eq. (27), for some constant e_1:

$$\dot{q}_3 = -\sqrt{2\xi \cos q_3 + e_1} \tag{27}$$

Zero dynamics phase portrait (Fig. 3) was obtained numerically, is locally stable and formed by closed curves.

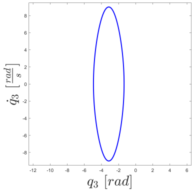


Figure 3: Zero dynamics

Overview

Existing robot being investigated in simulations

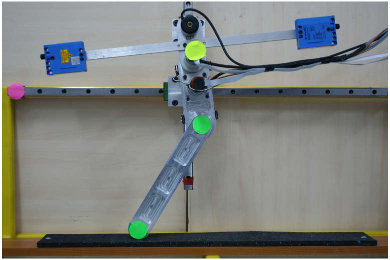


Figure 4: 3-link pendulum – experimental test-bed

3-link robot

- driven by *Maxon 200W EC-Powermax 30* brushless motors
- planetary gearhead of $N = 53$
- maximum torque of approximately 6 Nm

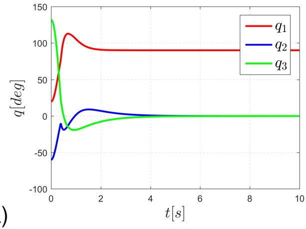
Table 2: Robot parameters

m_i – Mass $[\mathrm{kg}]$	Centre of mass $[\mathrm{m}]$	L_i – Length $[\mathrm{m}]$	Inertia $[\mathrm{kg\,m^2}]$
1.118	0.062	0.07	0.0118
1.593	0.074	0.15	0.0119
0.405	0.134	0.295	0.0117

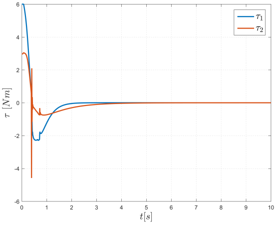
Simulation conditions

- the desired stabilization pose is the upright position for which the angles q_1^d, q_2^d and q_3^d were equal $90^\circ, 0$ and 0, respectively.
- initial condition: $q_{1_0} = 20^\circ$, $q_{2_0} = -60^\circ$ and $q_{3_0} = 131^\circ$ (exemplary one)
- the torque magnitude is restricted to 6 Nm – taken from existing robot
- simulation time $t = 10$ s.

The obtained angular trajectories are shown in Fig. 5a, while the control signal produced by motor is depicted on Fig. 5b.



a)



b)

Figure 5: a) Angular position of links, b) Motor torque c) Animation.

Thank You
For Your Attention